Solutions Mathematics 2 **January 2007**

1. $$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 2 & 5 & -3 & 13 \\ -3 & 4 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 7 & -1 & 13 \\ 0 & 1 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 20 & 20 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \text{ so } (x, y, z) = (3, 2, 1).$$

2. $$\begin{bmatrix} 1 & a & a \\ a & 4 & 2a \end{bmatrix} \sim \begin{bmatrix} 1 & a & a \\ 0 & 4 - a^2 & 2a - a^2 \end{bmatrix}$$

The system is inconsistent if $4 - a^2 = 0$ and $2a - a^2 \neq 0$, so if $a = -2$.

3. $$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{bmatrix}, \text{ so } A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

$$A^{-1}B^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{bmatrix},$$

so $(AB)^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$.

4. $\det B = \sqrt{b} \cdot \det \begin{bmatrix} b & 0 & 2 \\ 1 & 0 & 7 \\ 3 & 5 & 6 \end{bmatrix} = -5\sqrt{b} \cdot \det \begin{bmatrix} b & 2 \\ 1 & 7 \end{bmatrix} = -5\sqrt{b}\,(7b-2)$

   B is singular if $\det B = 0$, so if $-5\sqrt{b}\,(7b-2) = 0$, so if $b = 0$ of $b = \frac{2}{7}$.

5. $1 + 2 + 3 + \ldots + 999 + 1000 = \frac{1}{2} \cdot 1000 \cdot (1 + 1000) = 500500$.

6. Try $x_t = ct + d$, then $x_{t-1} = c(t-1) + d = ct - c + d$, and

   $x_{t-2} = c(t-2) + d = ct - 2c + d$.

   Substitution gives $ct + d = 3ct - 3c + 3d + 2ct - 4c + 2d + 4t - 3$,

   so $7c - 4d + 3 = 4ct + 4t = (4c+4)t$.

   Thus $7c - 4d + 3 = 0$ and $4c + 4 = 0$, so $c = -1$ and $d = -1$.

   So $x_t = -t - 1$ is a particular solution.

7. Sustitution gives $-1 = c + d$ and $2 = 4c + d$, so $c = 1$, $d = -2$.

   So $x_t = {x_{t-1}}^2 - 2 = f(x_{t-1})$, where $f(x) = x^2 - 2$.

   Then $f'(x) = 2x$, and $f'(-1) = -2 < -1$,

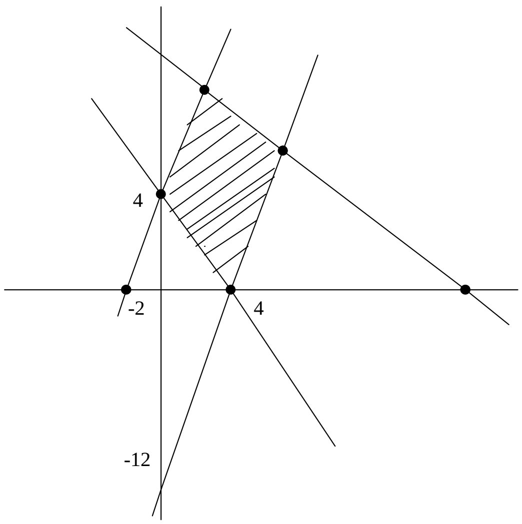
   and $f'(2) = 4 > 1$, so both equilibria are unstable.

8. Solve $(P - I)\mathbf{x} = 0$.

   $$\begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & \frac{1}{8} & 0 \\ \frac{1}{2} & -\frac{3}{4} & \frac{1}{8} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{4} & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{8} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{8} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{4} & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   so $x_1 = \frac{1}{2}x_3$ and $x_2 = \frac{1}{2}x_3$. Since $x_1 + x_2 + x_3 = 1$, we have $2x_3 = 1$, so $x_3 = \frac{1}{2}$, $x_1 = \frac{1}{4}, x_2 = \frac{1}{4}$.

Steady state $\mathbf{x} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{bmatrix}$.



9. Corner points $(4,0), (6,6), (2,8), (0,4)$; $z = x_1 - x_2$.

$z(4,0) = 4,\ z(6,6) = 0,\ z(2,8) = -6,\ z(0,4) = -4$

so min $z(2,8) = -6$.

10.
$$\begin{array}{ccccccc|c} 1 & -1 & -2 & -3 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 1 & 1 & 0 & 0 & 9 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 5 \\ 0 & 2 & 0 & 1^* & 0 & 0 & 1 & 4 \end{array}$$

$$\begin{array}{ccccccc|c} 1 & 5 & -2 & 0 & 0 & 0 & 3 & 12 \\ \hline 0 & -3 & 1 & 0 & 1 & 0 & -2 & 1 \\ 0 & -1 & 1^* & 0 & 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 1 & 0 & 0 & 1 & 4 \end{array}$$

$$\begin{array}{ccccccc|c} 1 & 3 & 0 & 0 & 0 & 2 & 1 & 14 \\ \hline 0 & -2 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 1 & 0 & 0 & 1 & 4 \end{array}$$

so max $z(0,1,4) = 14$.

Shadow prices $y_1 = 0$, $y_2 = 2$, $y_3 = 1$.

Other final tables are possible, and $(y_1,\ y_2,\ y_3) = (\alpha, 2-\alpha, 1-\alpha)$ is an optimal solution for the dual for all α such that $0 \le \alpha \le 1$.

11. $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ is in the core if

$$\begin{array}{ccccccccccccc}
 & & & & c_1 & \ge & 5 & & & & & & \\
 & & & & c_2 & \ge & 4 & & & & & & \\
 & & & & c_3 & \ge & 7 & & & & & & \\
 & & c_1 & + & c_2 & \ge & e & \rightarrow & c_3 & \le & 18 & - & e \\
 & & c_1 & + & c_3 & \ge & 14 & \rightarrow & c_2 & \le & 18 & - & 14 & = & 4 \\
 & & c_2 & + & c_3 & \ge & 12 & \rightarrow & c_1 & \le & 18 & - & 12 & = & 6 \\
c_1 & + & c_2 & + & c_3 & = & 18 & & & & & &
\end{array}$$

Thus $c_2 = 4$ and $18 = c_1 + c_2 + c_3 \le 18 - e + 4 + 6 = 28 - e$.

So $e \le 28 - 18 = 10$.

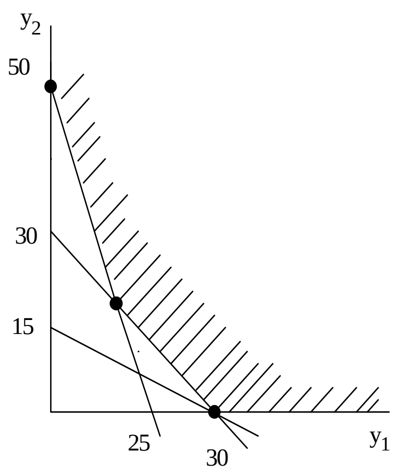
If $e = 10$, then $c_1 = 6$ and $c_3 = 8$.

Then $c = \begin{bmatrix} 6 \\ 4 \\ 8 \end{bmatrix}$ is indeed in the core.

so the largest possible e equals 10.

12. $b_1 + b_2 = \begin{bmatrix} 50 \\ 40 \end{bmatrix}$

$$\begin{array}{llrcrcr}
\text{min.} & w = 50y_1 & + & 40y_2 & & \\
\text{s.t.} & 2y_1 & + & 2y_2 & \geq & 60 \\
 & 6y_1 & + & 3y_2 & \geq & 150 \\
 & 3y_1 & + & 6y_2 & \geq & 90 \\
 & y_1 & \geq 0 & y_2 & \geq & 0
\end{array}$$



$w(30, 0) = 1500$, $w(20, 10) = 1400$, $w(0, 50) = 2000$

so min $w(20, 10) = 1400$ and $y_1 = 20$, $y_2 = 10$

"Shadow values": player $1 : 20 \cdot 30 + 10 \cdot 10 = 700$

player $2 : 20 \cdot 20 + 10 \cdot 30 = 700$

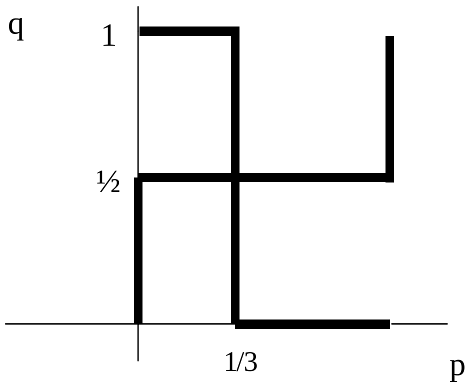
Owen vector $\begin{bmatrix} 700 \\ 700 \end{bmatrix}$

13. Row 1 is better than row 2 if $5q + 1 - q > 2q + 4(1 - q)$, so if $4q + 1 > 4 - 2q$, so if $q > \frac{1}{2}$.

Column 1 is better than column 2 if $-5p - 2(1 - p) > -p - 4(1 - p)$, so if $-3p - 2 > 3p - 4$, so if $p < \frac{1}{3}$.

Nash equilibrium $(\mathbf{p}, \mathbf{q}) = \left(\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right)$

$$v(A) = \mathbf{p}^{\intercal} A \mathbf{q} = \begin{bmatrix} \tfrac{1}{3} & \tfrac{2}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3.$$



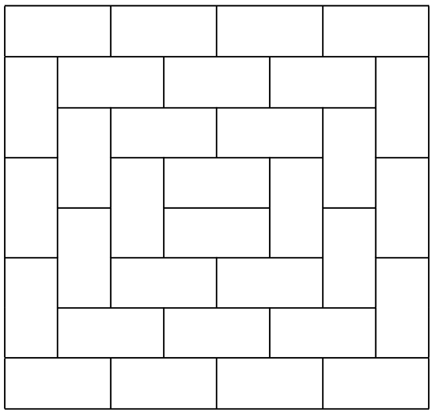
14.

$$\begin{bmatrix} 2 & 2 & 4 & 7 & 8 & 4 \\ 3 & 2 & 5 & 5 & 5 & 5 \\ 4 & 2 & 6 & 5 & 4 & 4 \\ 2 & 3 & 4 & 4 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} -2 \\ -2 \\ -2 \\ -2 \\ \\ \\ \end{array} \qquad \begin{array}{c} \begin{bmatrix} 0^* & 0^* & 2 & 5 & 6 & 2 \\ 1^* & 0^* & 3 & 3 & 3 & 3 \\ 2^* & 0^* & 4 & 3 & 2 & 2 \\ 0^* & 1^* & 2 & 2 & 2 & 1 \\ 0^* & 0^* & 0^* & 0^* & 0^* & 0^* \\ 0^* & 0^* & 0^* & 0 & 0^* & 0^* \end{bmatrix} \\ \begin{array}{cccccc} +1 & +1 & & & & \end{array} \end{array} \begin{array}{c} -1 \\ -1 \\ -1 \\ -1 \\ \\ \\ \\ \end{array}$$

$$\begin{array}{c} \begin{bmatrix} 0^* & 0^* & 1 & 4 & 5 & 1^* \\ 1^* & 0^* & 2 & 2 & 2 & 2^* \\ 2^* & 0^* & 3 & 2 & 1 & 1^* \\ 0^* & 1^* & 1 & 1 & 1 & 0^* \\ 1^* & 1^* & 0^* & 0^* & 0^* & 0^* \\ 1^* & 1^* & 0^* & 0^* & 0^* & 0^* \end{bmatrix} \\ \begin{array}{cccccc} +1 & +1 & & & +1 & \end{array} \end{array} \begin{array}{c} -1 \\ -1 \\ -1 \\ -1 \\ \\ \\ \\ \end{array} \qquad \begin{bmatrix} 0^* & 0 & 0 & 3 & 4 & 1 \\ 1 & 0^* & 1 & 1 & 1 & 2 \\ 2 & 0 & 2 & 1 & 0^* & 1 \\ 0 & 1 & 0 & 0 & 0 & 0^* \\ 2 & 2 & 0^* & 0 & 0 & 1 \\ 2 & 2 & 0 & 0^* & 0 & 1 \end{bmatrix} \qquad \begin{array}{l} \\ P_1 - A \\ P_2 - B \\ P_3 - E \\ P_4 - F \\ \\ \end{array}$$

15.

a)



b) A strategy for player two that prevents player one from winning is to put an O in the brick in which player one just put an $\times$.